

Arithmetic by Counters

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1 Introduction

Before electronic computers came into use, the word *computer* referred to a person who performed arithmetic manually. Though Arabic notation for numbers and pen-and-paper algorithms for computing with them were known to European mathematicians in the later centuries of the SCA period, people with more mundane commercial occupations used Roman numerals and the abacus up until the seventeenth and eighteenth centuries.

In this article, I'll give a description of arithmetic using counters on a board, which seems to have been the usual way of performing arithmetic in Europe throughout the SCA period. I've relied heavily on Pullan [3] for photographs of abacus-related items kept in European museums. Background information on the history of mathematics is based on Cajori [2] and Yeldham [5].

2 The Flat-Board Abacus

The type of abacus used in mediaeval Europe was not the bead-frame abacus still used in some Eastern countries today. Europeans used a flat surface divided into a number of horizontal or vertical lines, upon which counters were placed. The abaci that I have made use vertical divisions, so for convenience I will write as if the abacus is made up of columns. A horizontal abacus works the same way, everything is just turned about ninety degrees.

The layout of the abacus is familiar to anyone who knows Arabic notation. The right-most column represents the number of ones; the second right-most column represents the number of tens; the third the number of hundreds, etc. In fact, the tenth-century monk Gerbert (later Pope Sylvester II) used counters marked with Arabic digits, and placed a counter bearing the appropriate digit in each column to represent a number. It was more usual, however, to have a large supply of identical counters, and

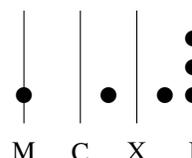


Figure 1: 1058 (MLVIII) on an abacus.

a number of counters equal to the number of ones, tens, etc., was placed in the appropriate column.

To make the abacus easier to read, and to reduce the number of counters required, a counter placed between two columns was used to represent five counters in the column to its right. For example, eight is represented by three counters in the right-most column and one counter between the first and second columns. Using this technique, we can relate the abacus to Roman notation: the numerals I, X, C and M represent counters placed on the columns, while the numerals V, L and D represent counters placed between the columns. Some abaci, in fact, have the columns labelled with Roman numerals to make them easier to read; others have every third column marked by a cross (×) in the same way that we use commas or spaces in writing large numbers with Arabic notation.

If you understand Arabic notation, and have grasped the layout of the abacus, you will see how integer arithmetic can be done on the abacus. A genuine period exposition in English can be found in the Second Dialogue of Robert Recorde's *The Grounde of Artes* [4]. Recorde's algorithms for the basic arithmetic operations are more or less the same as the pen-and-paper algorithms I was taught in primary school. I'll describe the multiplication algorithm here as an example of how the abacus was used.

The numbers to be multiplied are the *multiplier* and the

multiplicand; of course it doesn't matter to the result (the *product*) which is which, but they play different roles in the algorithm so we have to give them names.

We start by setting the multiplier at the bottom of the abacus, and setting out the multiplicand above it. A line is drawn across the abacus to separate them, and another line to separate the product from the multiplier, though leaving a space will do if you don't want to draw on your abacus.

We work on one column of the multiplicand at a time, say, the n th column. Recorder works from top to bottom on a horizontal abacus, which is equivalent to working left to right on a vertical abacus, but working right to left works equally well.

For each counter in the n th column of the multiplicand, set down a copy of the multiplier at the top of the abacus as if the n th column was the first column, i.e. shift the copy $n - 1$ columns to the left. You can keep count by removing the counter from the multiplicand that produced the copy.

If there is a counter in the space to the left of the column, you could copy the multiplier into the n th column five times. A more efficient method, however, is to copy half the multiplier into the $(n + 1)$ th column, rounding down if the multiplier is odd. If the multiplier is odd, also put a counter in between the n th and $(n + 1)$ th column (this counter is equivalent to half a counter in the $(n + 1)$ th column).

If, like me, you were made to invest much of your childhood schooling in memorising multiplication tables, you can speed up the computation by replacing the repeated addition I've just described with a single multiplication. Recorder uses multiplication tables in his pen algorithms, but the average abacus-user was an illiterate who never went to primary school, and would not have known the tables.

Having set a large number of counters onto your abacus, you will probably want to clean them up by adding them all into a single number. To add, bring all of the counters in each column together. For every five counters in a column, replace them by one counter between that column and the next, and for every two counters between columns, replace them with one counter in the column to the left. Of course you can keep putting down more and more counters for so long as you have counters and space, if you like, but you *will* have to add them all together some

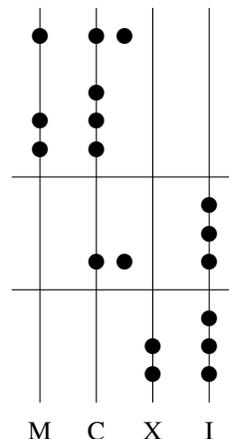


Figure 2: Multiplying 153 (multiplicand) by 23 (multiplier), after the first two columns have been multiplied. The partial products are 23×100 and $23 \times 50 = 11\frac{1}{2} \times 50$.

time.

At the end, after all of the columns in the multiplicand have been multiplied in this way, and all of the results have been summed, the product will be left in the space above the multiplicand. It is straightforward to prove that the algorithm is correct using the Distributive Law.

Arithmetic involving non-integers is cumbersome using the period techniques due to the way such numbers were represented and I've never seen a good description of how it was done. The fractional part of the number was represented by a sum of fractions in inconvenient denominations such as $\frac{1}{2}$ and $\frac{1}{3}$ that went by their Latin names rather than symbols. Each fraction had its own column in the abacus, either to the right of the integer part, or in another row of columns below it.

The techniques used by modern computers for dealing with non-integers can be adapted to the abacus but this article is about mediaeval computing and we won't talk about these methods here.

3 Constructing an Abacus

A flat-board abacus is simply a set of parallel lines on a flat surface, and mathematics historians give descriptions

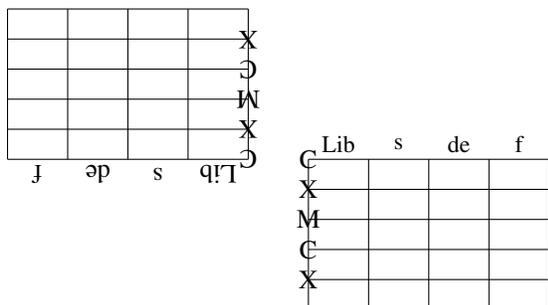


Figure 3: The layout of the reckoning table at Strasbourg. The abaci can be used vertically for working with currency, or horizontally for working with plain numbers.

of many ways of achieving this based on their interpretation of contemporary accounts. In a pinch, you can simply draw lines in the dirt and use pebbles for counters.

According to Barnard [1] (as quoted by Pullan),

the extreme rarity of specimens of the counting board is remarkable, considering its general vogue in Western Europe during the six centuries from 1200 to the French Revolution, as is the survival of any examples of its more perishable substitute the reckoning cloth ... The [three] tables at Basle, the doubtful one at Nuremberg and the [five] reckoning cloths at Munich, are all that I have been able to find after considerable search and enquiry...

Pullan, however, seems to have discovered another one in the Strasbourg Municipal Museum, apparently from the end of the sixteenth century, of which he has a photograph. It is a wooden table with a lip around the edge (presumably to stop unused counters from falling off). Two abaci are painted on top of it, one on each side of the table, as shown in Fig. 3.

A cheaper and more portable alternative is to use a cloth. I made one for a collegium by taking a rectangular piece of fabric and using fabric paint to draw an abacus pattern on it. The pattern I used is a slightly simplified version of a diagram in Yeldham's book showing a twenty-seven column abacus with the columns grouped in threes by semi-circles at the top. Yeldham says this

abacus was used in England in 1111; the pattern is very similar to the one reported in Cajori's book as being described by Bernelinus, a pupil of Gerbert. My abacus is shown in Fig. 4. In hindsight, I think I should have used only nine or twelve columns because the current columns are too narrow, and I have no use for twenty-seven-digit numbers. I can only speculate what use Gerbert had for them.

For counters, there are many references to the use of pebbles for computation in classical literature; Pullan gives a summary of these in an appendix. In later times, counters were minted from metal in the same way as coins, and Pullan devotes a whole chapter to these (he calls them "jettons") and has many photographs of extant examples. Modern coins seem a good substitute for those of us who don't own a mint.

4 Conclusion

A classmate of mine once complained of a complex analysis technique in electrical engineering, "What do we need to learn this for? The computer can do it." Of course there's no future in that line of argument if you design computer systems. Studying methods of computation other than the ones I learnt mechanically at school has improved my feeling for how computation works, and even the most mundane computations are now accorded their due respect.

References

- [1] F. P. Barnard, *The Casting-Counter and the Counting-Board*, 1916.
- [2] F. Cajori, *A History of Mathematics*, 5th Ed., 1991.
- [3] J. M. Pullan, *The History of the Abacus*, 1968.
- [4] R. Recorde, *The Grounde of Artes*, 1542.
- [5] F. A. Yeldham, *The Story of Reckoning in the Middle Ages*, 1926.

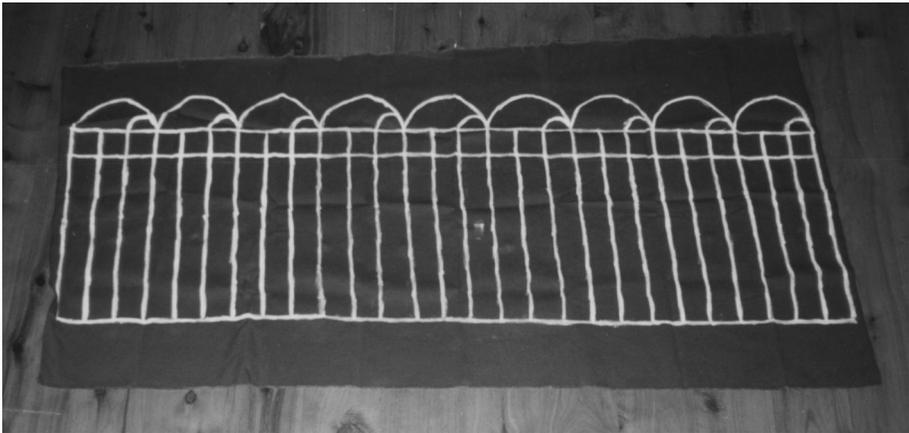


Figure 4: My reckoning cloth