The World: Just How Big Is It?

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1 Introduction

The shape of the world is one of my favourite examples of how "common sense" can be hopelessly misleading.

My, and probably your, common sense tells us that the word is flat. It looks flat to a person standing on its surface, and we usually navigate as if the world is a flat sheet. (If you don't believe me, please send me an e-mail detailing the spherical geometry that you used when calculating your walk or drive to work.)

Of course, we learn in school that the world is actually spherical, and that it only appears flat because its enormity makes its curvature too shallow to be apparent to miniscule beings like us. But how would you demonstrate this to someone who insisted that it was "common sense" that the world was flat?

These days, given enough money, you could take your doubter on a trip around the world by ship, aeroplane or spaceship. Despite popular modern images of a mediaeval world populated entirely by pre-Colombus flat-earthers, however, many ancient and mediaeval thinkers found ways to estimate the size and shape of the Earth without access even to ocean-going ships, let alone aeroplanes or spacecraft.

2 Aristotle

Aristotle (384 - 322 BC), one of the most influential thinkers in the history of Western thought, addresses the size, shape and motion of the Earth in *On the Heavens*. In Part 13, he describes and refutes the views of various other philosophers who say that the Earth is flat, or cylindrical, or infinitely big. In Part 14, he describes and justifies his own opinion that the Earth is a very large sphere, albeit small compared to the size of the universe.

He first argues that, since the substance of the Earth is observed to fall towards the centre of the universe (that is, Earth, in Aristotle's cosmology), the Earth's mass be must equally distributed about its centre. Mass distributed in any other way would over-balance and fall closer to the centre.

More practically, he observes that heavenly bodies appear to move in a circle about the Earth: if the Earth were flat, the sun, moon and stars would instead move in a straight line across the sky. Furthermore, the horizon moves as one travels north or south, so that it is possible to see stars in Egypt and Cyprus that are not visible from further north (where they are below the southern horizon). From a flat earth, any star above the plane of the Earth's surface would be visible everywhere.

Finally, he says that "those mathematicians who try to calculate the size of the earth's circumference arrive at the figure $400,000$ stadia". He does not say who these mathematicians were, however, or how they arrived at the figure of $400,000$ stadia. Modern histories of the circumference of the Earth say that Archimedes (c. 287 - c. 212 BC) later proposed $300,000$ stadia, but none of them are any clearer as to where this figure comes from.

3 Eratosthenes

Eratosthenes (c. 276 - c. 194 BC), the chief librarian of Alexandria, made what is probably the most famous attempt to establish the size of the Earth in antiquity. I was first introduced Eratosthenes' method as an exercise in high school geometry, but you can also read about it in histories of science like Robert Crease's *The Prism and the Pendulum* (2003). Eratosthenes' own *On the Measurement of the Earth* is lost but we know of his work through a description given by Cleomedes in *Caelestia*.

Eratosthenes reasoned that, if he could work out both the distance along the surface of the Earth between two points and the portion (angle) of the Earth's circumference that you needed to traverse in order to travel between

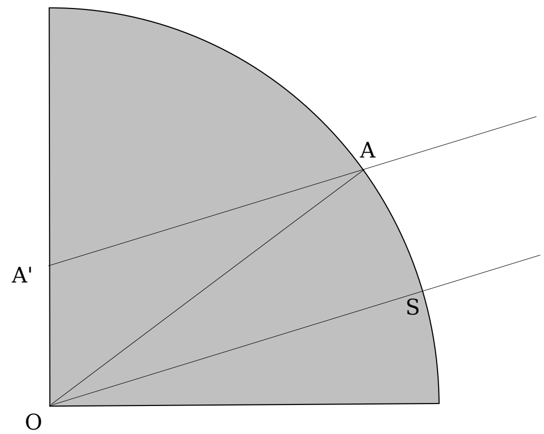

Figure 1: Erastosthenes' method for calculating the circumference of the Earth.

the two, he could calculate the circumference of the Earth by multiplying the distance between the two points by the number of times that the angle between them fitted into a full circle (that is, 360 degrees).

Of course Eratosthenes wasn't able to place a giant protractor at the centre of the Earth to measure the angle between two points on the Earth's surface. Instead, he used the change in the angle of the sun's rays when seen at two different latitudes together with some elementary geometry to estimate the angle that he actually needed to measure.

Figure 1 shows the idea. Eratosthenes noted that the sun shone directly overhead at noon on the solstice in the city of Syene (S), now Aswan in southern Egypt. At the same time of year in Alexandria (A), which was nearly (but not quite) directly to north, the sun cast shadows at an angle equivalent to one-fiftieth of a full circle (that is, 7.2 degrees).

Assuming the distance between the Earth and the sun to be very much greater than the distance between Alexandria and Syene, the sun's rays at Alexandria (line AA' in the diagram) are very nearly parallel to the rays at Syene, which point directly at the centre of the Earth O. One of Euclid's theorems concerning parallel lines then tells us that the angle AOS between Alexandria and Syene is equal to the angle of the sun's rays observed at Alexandria at solstice.

Eratosthenes estimated the distance between Alexandria and Syene to be $5,000$ stadia. Since the foregoing observations indicate that this distance represents about one-fiftieth of the circumference of the Earth, the full circumference is about $250,000$ stadia. Unfortunately we do not know exactly how long Eratosthenes' "stadium" was, since the length of a stadium differed from city to city. Depending on which stadium he used, $250,000$ stadia comes to between $39,250$ and $52,250$ kilometres.

Cleomedes doesn't relate how Eratosthenes came by the "5000 stadia" from Alexandria to Syene, and modern writers give differing accounts: Crease says he obtained it from the "royal surveyors"; Al-Khalili (2010) suggests that he commissioned someone to walk from Alexandria to Syene, counting the number of steps taken; and Rubin (2011) asserts that he obtained it from an estimate of the average speed of a camel caravan travelling between the two cities. Other writers (including Al-Khalili) suggest that he may have even obtained the figure from a previous calculation of the circumference of the Earth, making the new calculation circular.

4 The House of Wisdom

A recent book by Jim Al-Khalili (2010) relates that Abu Ja'far Abdullah al-Ma'mun (786-833), the Abbasid caliph of Baghdad from 813 until his death, established *Bayt al-Hikma* ("The House of Wisdom") in Baghdad in order to "collect all the world's books under one roof." Along with numerous other adventures in astronomy, mathematics and geography, al-Ma'mun is said to have set his astronomers the task of re-creating Eratosthenes' experiment under more controlled conditions.

Al-Ma'mun's astronomers used the plains of Sinjar, about 100 kilometres west of Mosul (now in Iraq). One group of astronomers walked due north, counting their steps as they went and marking the distance with arrows, until they had travelled one degree of latitude as determined by the positions of the stars. Another group walked south and measured similarly. The astronomers double-checked the measurements on their return journey, and, by taking the average of all of the measurements, they arrived at a distance of 56.6 Arabic miles (about 109 kilometres) for one degree of the Earth's surface. This gives an estimate of $39,224$ kilometres for the circumference of

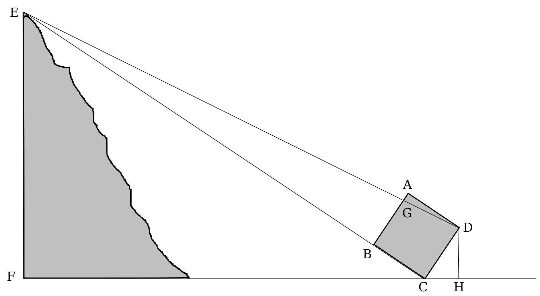

Figure 2: Al-Biruni's method for calculating the height of a mountain, adapted from Al-Khalili's book.

the Earth.

5 Al-Biruni

Abu Rayhan Muhammad ibn Ahmad al-Biruni's *Kitab Tahdid al-Amakin* ("The Determination of the Coordinates of Cities") reports that one of al-Ma'mun's leading astronomers, Sanad ibn Ali al-Yahudi, later proposed another method for determining the size of the Earth. Sanad observed that it should, in principle, be possible to determine the radius of the Earth by climbing to the top of a mountain from which the sea is visible. The climber can measure the angle between a plumb line and the line to the horizon as seen from the mountain, and combine this with knowledge of the height of the mountain and some trigonometry to compute the radius of the Earth.

Sanad doesn't seem to have actually carried out the experiment that he proposed, and it isn't immediately obvious how one might measure the height of the mountain. Al-Biruni himself, however, proposed and carried out a more complex scheme that first determined the height of a mountain, and from this he was able to work out the radius of the Earth.

Figure 2 illustrates al-Biruni's method for determining the height of a mountain using a square board, called $ABCD$ in the diagram. Al-Biruni's board was one cubit one each side, but any square will do. (The one shown in Figure 2 is much larger than any practical square, of course.)

Holding the board vertically, place one corner (C) of the board on the ground at sea level, and rotate the board until the bottom edge (BC) aligns with the top of the mountain to be measured. Attach a straight edge to point D on the board, and rotate it until this edge is also aligned with the top of the mountain. This edge will intersect the side AB of the square at some point, called G in the diagram.

Using a little geometry from Euclid, we can see that the triangles DAG and ECD are *similar*, that is, have the same three internal angles and are therefore magnifications of one another. Since we can measure DAG directly, we can work out the distance between C and the top of the mountain (E) by multiplying the side of the square (AD) by the ratio $\frac{CD}{AG}$.

Having worked out the length CE this way, drop a plumb line from the corner D to the ground (point H). Assuming that the curvature of the Earth between the board and the mountain is negligible, we can see that the triangles DHC and CFE are also similar. Using our knowledge of the length CE and triangle DHC, we can work out the height of the mountain (EF) by multiplying the length CH by the ratio $\frac{CE}{CD}$.

Armed with the height of the mountain, we can finally implement Sanad's idea, illustrated in Figure 3. At the top of the mountain (A), measure the angle OAT between a plumb line (OA) and the horizon (T).

Since the angles of a triangle must add up to 180 degrees, and the angle (ATO) between a radial line and a tangent is 90 degrees, we see that $OAT + TOA + 90 = 180$. We have just measured OAT, so we can use this simple equation to compute the angle TOA, made at the centre of the Earth by the radial lines to the mountain and the horizon.

If we know our trigonometry (as al-Biruni did), we will further see that the cosine of the angle TOA is the ratio $\frac{OT}{OA}$. Assuming that the Earth is perfectly spherical, the distance OA is equal to $OT+BA$, that is, the radius of the Earth plus the height of the mountain. So, our trigonometry tells us that $\cos(TOA) = \frac{OT}{OT+BA}$. Since we know the angle TOA and the height of the mountain BA, we can solve the equation for the sole unknown distance OT, the radius of the Earth.

Aside from its complexity, the chief difficulty with this method is in measuring the distance AG when determining the height of the mountain. Since any practical square $ABCD$ is very small compared to any mountain worthy

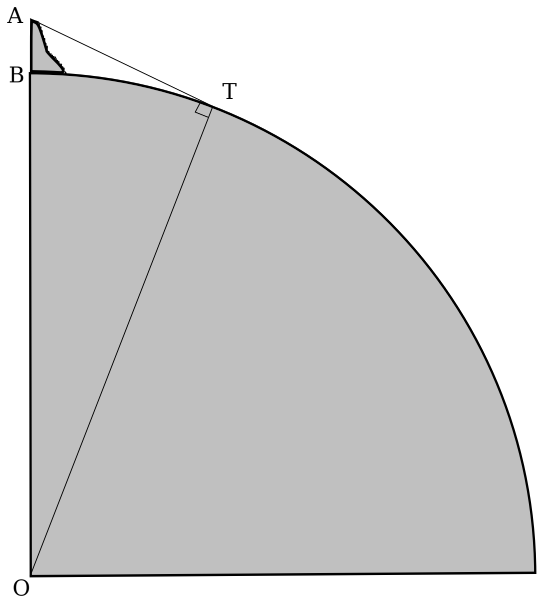

Figure 3: Sanad's method for calculating the radius of the Earth, as used by al-Biruni.

of the name, the lines CE and DE will be very nearly parallel, and the distance AG will be very small. If you are standing one kilometre from the top of the mountain with a $1\text{m} \times 1\text{m}$ board, for example, the distance AG will be only one millimetre. A hundred-metre difference in the height of the mountain will translate to a difference of less than a tenth of a millimetre in the position of G.

Al-Biruni, nonetheless, computed a value of $12,803,337$ cubits for the radius of the Earth. Unfortunately we aren't certain exactly how long his cubit was, but his value comes to between $6,162$ and $6,386$ kilometres for the lengths of a cubit suggested by Wikipedia's definition of an Arab mile.

6 My Own Attempt

I live in the Shire of Adora, which lies on the Australian coast between the Illawarra Escarpment and the Tasman Sea. A modern-day al-Biruni, then, should be able to estimate the size of the Earth by measuring the height above sea level of a point at the top of the escarpment, and then

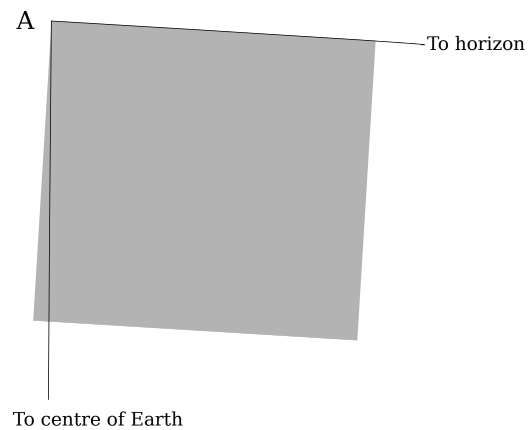

Figure 4: My apparatus for measuring the angle of the horizon.

measuring the angle of the horizon when seen from the same point.

I wasn't able to achieve the precision required to determine the height of a mountain using al-Biruni's method. After a few attempts produced absurd results for the height a few nearby mountains, I concluded that I'd need to obtain a measurement produced by a professional surveyor with better equipment than I was able to make.

I tried measuring the angle to the horizon from three different points on the Illawarra Escarpment, slightly refining my equipment and technique each time. I took the last, and hopefully most refined, measurement from the summit of Mt Keira, which overlooks the centre of Wollongong. According to signage erected by the New South Wales National Parks and Wildlife Service, the summit rises 464 metres above sea level. A taller mountain would be better, but none of the mountains convenient to me are much higher than Mt Keira.

Using a plumb line and a 270mm set square from my technical drawing class, I aligned the top edge of the square with the horizon and dropped the plumb line from the top corner nearest me, as shown in Figure 4. This forms a right-angled triangle with the plumb line as its hypotenuse, the left-hand edge of the square as one side, and the bottom edge of the square as the other side. The length of the last edge is what I need to measure in order to calculate the size of the Earth.

I repeated this procedure several times, using different points on the horizon each time, resulting in measurements between 4.5mm and 6mm. This variation suggests that my procedure is not very reliable, probably because my hand-eye co-ordination wasn't very precise in aligning the top edge with the horizon. For the sake of illustrating the calculation, however, let's say my average measurement was 5.5mm.

Since the leftwards edge of the square in Figure 4 is parallel to the line OT in Figure 3, and the plumb line is aligned with OA, the angle made by the plumb line and the left-hand edge of the square is the same as the angle TOA made by the plumb line and the horizon at the centre of the Earth. I can therefore derive $\cos(TOA)$ from my triangle using Pythogoras' theorem and the definition of cosine by

$$\cos(TOA) = \frac{0.270}{\sqrt{0.270^2 + 0.0055^2}}.$$

(Note that I've converted the millimetres into metres.)

We can now use Sanad's equation and the height of Mount Keira to estimate of the radius of the Earth OT:

$$\begin{aligned} \frac{OT}{OT+464} &= \frac{0.270}{\sqrt{0.270^2+0.0055^2}} \\ \sqrt{0.270^2 + 0.0055^2}OT &= 0.270(OT + 464) \\ (\sqrt{0.270^2 + 0.0055^2} - 0.270)OT &= 0.270 \times 464 \\ OT &= \frac{0.270\times464}{\sqrt{0.270^2+0.0055^2}-0.270} \\ OT &= 3113936.2 \end{aligned}$$

According to this calculation, the radius of the Earth is just $3,114$ kilometres!

7 Other Modern Measurements

NASA's *Earth Fact Sheet* (Williams, 2010) puts the radius of Earth at 6356.8 kilometres at the poles and 6378.1 kilometres at the equator. This equates to a circumference of $39,940$ kilometres around the poles and $40,075$ kilometres around the equator.

In fact, the metre was originally chosen to represent one ten-millionth of the distance between the Equator and the poles, or "the ten millionth part of one quarter of the terrestrial meridian", as French Metrology (n.d.) puts it. If the Earth were a perfect sphere, and post-Revolutionary French metrologists perfect observers, the full circumference of the Earth would be $40,000,000$ metres.

8 Conclusion

Standing at the top of Mount Keira with my 270mm set square, I ought to have measured a displacement of about 3.25mm. This is large enough to be measurable, but I presumably failed to align my square with the horizon with sufficient accuracy. Better equipment, a steadier hand and a taller mountain would all help in getting a more accurate measurement. Indeed, al-Biruni's figure for the Earth's radius is within a few hundred kilometres of NASA's value.

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